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Towards a theory of granular plasticity

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Abstract. A theory of granular plasticity based on the time-averaged rigid-plastic flow equations is presented. Slow granular flows in hoppers are often modeled as rigid-plastic flows with frictional yield conditions. However, such constitutive relations lead to systems of partial differential equations that are ill-posed: they possess instabilities in the short-wavelength limit. In addition, features of these flows clearly depend on microstructure in a way not modeled by such continuum models. Here an attempt is made to address both short-comings by splitting variables into 'fluctuating' plus 'average' parts and time-averaging the rigid-plastic flow equations to produce effective equations which depend on the 'average' variables and variances of the 'fluctuating' variables. Microstructural physics can be introduced by appealing to the kinetic theory of inelastic hard-spheres to develop a constitutive relation for the new 'fluctuating' variables. The equations can then be closed by a suitable consitutive equation, requiring that this system of equations be stable in the short-wavelength limit. In this way a granular length-scale is introduced to the rigid-plastic flow equations.

Key words: granular flow, granular temperature, plasticity

1. Introduction

The flow and handling of granular materials is of major importance to many industries. Yet, despite efforts over several decades, the modeling of such flows has achieved only modest success. Dense gravity-driven flows in hoppers have been often modeled as elastic–plastic continua, for example. In this picture, the granular material flows as a plastic with a frictional yield condition, and deforms as an elastic solid otherwise. This model has been used to analyze mass flows, where the material is flowing throughout the hopper, but has failed to provide adequate agreement with experiment in the prediction of quantities such as discharge rate, for example [1]. Despite such shortcomings, Jenike's [2] construction of steady-state incompressible rigid-plastic radial solutions in hoppers with simple geometries has been of considerable importance in hopper design [3, Chapter 10]. These are solutions for quasistatic flow (inertial effects are neglected) where grains travel radially in conical or wedge-shaped hoppers. Only recently have numerical solutions of steady-state quasistatic flows in more complicated geometries been produced [4].

However, it is now clear that there are serious mathematical difficulties with the equations for time-dependent incompressible rigid-plastic flow (IRPF). In many instances, the equations for such flows have been shown to be ill-posed, *i.e.*, they possess instabilities with arbitrarily short wavelengths [5,6]. Hence, it is problematic to interpret the steady-state rigid-plastic flow equations as long-time solutions of the time-dependent equations. Additionally, both the steady-state and time-dependent equations have physical shortcomings. There are several features of hopper flows where the particle size may be important, such as chute flows [7] and shear-banding [8] indicating that it may not be appropriate to model such dense granular flows as continua.

Nonetheless, attempts have been made to describe such flow features in continuum theories in some average sense [9]. Indeed, the development of the shear-banding instability may be related to ill-posedness in continuum plastic flow models [10]. One approach to dealing with this in a continuum theory is to model materials with internal structure by the inclusion of extra terms motivated by physics at the granular scale. These extra terms can damp the instabilities that lead to ill-posedness and can be used to predict shear-band thickness Muhlhaus [9]. Including such damping terms which act at a granular length-scale and 'thickening' the shear-bands is a practical approach to studying granular flows using continuum equations.

Indeed, such an approach is analogous to including Reynolds stresses in the Navier–Stokes equations to model turbulence. In granular systems the analog of the Reynolds stresses is granular temperature. Granular temperature arises naturally in the theory of inelastic gases [11]. Savage [12] introduced granular temperature into plastic flow equations in order to facilitate the introduction of the particle size into the continuum equations for plastic flow. We will introduce granular temperature for a similar purpose here, appealing to the kinetic theory of inelastic gases to introduce grain size into the equations.

We begin by examining the IRPF equations in two-dimensions and review the proof that these equations are ill-posed. We then introduce a granular temperature in a general way, before building it specifically into the plastic flow equations. We then show that with an appropriate choice of constitutive relations, the modified IRPF equations are well-posed.

2. Equations for an incompressible rigid-plastic flow

We begin by considering the flow under gravity of an incompressible granular material in a wedge-shaped hopper under plane strain. We model the flow of this material as a continuum rigid-plastic flow. The equations for such a flow consist of the incompressibility condition,

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where u_i are the components of the velocity field, and the momentum equations:

$$\rho\left(\frac{\partial u_i}{\partial t} + \boldsymbol{u}\cdot\nabla u_i\right) + \nabla_j \sigma_{ij} = \rho g_i, \tag{2}$$

where σ_{ij} is the symmetric stress tensor, g_i is the acceleration due to gravity and ρ is the density of the flowing granular material. Note that we define σ to be positive when forces are compressive.

Here we will consider flows where plastic deformation is occurring everywhere in the hopper, *i.e.*, the material is at plastic yield throughout the hopper. We will use an extended von Mises yield condition. In terms of the principal stresses σ_i, this condition is written as

$$(\sigma_1 - \sigma)^2 + (\sigma_2 - \sigma)^2 + (\sigma_3 - \sigma)^2 \le k^2\sigma^2, \tag{3}$$

where $k = \sqrt{2}\sin\varphi$, φ is the internal angle of friction of the material and $\sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ is the average trace of the stress tensor (we will refer to σ as the average stress). If this inequality is satisfied exactly then the material is deforming plastically. Under plane strain $\sigma_2 = \sigma = \frac{1}{2}(\sigma_1 + \sigma_3)$, so in this case we may consider a strictly two-dimensional yield condition:

$$(\sigma_1 - \sigma)^2 + (\sigma_3 - \sigma)^2 = k^2\sigma^2. \tag{4}$$

We now assume a *non-associated flow* rule of the form

$$\sigma_{ij} = \sigma\delta_{ij} + \mu V_{ij}, \tag{5}$$

where $V_{ij} = \nabla_{(i}u_{j)} = (\nabla_i u_j + \nabla_j u_i)/2$ and μ is some, as yet unspecified, scalar function of the normal stress and strain rates. If we compare the flow rule (5) to the yield condition (3), then we see such a flow will satisfy the yield condition exactly if we choose the function μ to be

$$\mu = \frac{k\sigma}{\|V\|}, \tag{6}$$

where $\|V\| = (V_{ij}V_{ji})^{1/2}$.

Equations (1), (2), (5) and (6) form a closed system for incompressible rigid-plastic flow in plane strain. For granular flows in hoppers, these equations are only valid for so-called mass flows where the material is flowing throughout the hopper. When the hopper is not sufficiently steep, funnel flows can develop where material flows down a central funnel leaving a stagnant region adjacent to the walls. Indeed, radial solutions have been used to study the transition between mass and funnel flow which is thought to occur when the rigid-plastic equations become singular as the rate of deformation vanishes [13]. We will restrict our attention to mass flows where rigid-plastic flow occurs everywhere in a given domain.

Combining Equations (1), (2), (5) and (6) together we obtain the equations:

$$\left(\frac{\partial u_i}{\partial t} + \boldsymbol{u}\cdot\nabla u_i\right) = \rho g_i - \nabla_j\left(\sigma\left(\delta_{ij} - kA_{ij}\right)\right), \tag{7}$$

$$\nabla_i u_i = 0, \tag{8}$$

where $A_{ij} = V_{ij}/\|V\|$. We note that $\mathrm{Tr}(A_{ij}) = 0$ and that $\mathrm{Tr}(A_{ik}A_{kj}) = 1$.

The Equations (7) and (8) have been shown to be linearly ill-posed in certain geometries and for certain parameter values [5]. Specifically, the linearized equations of motion that describe the propagation of a small disturbance in the flow, possess unstable plane-wave solutions in the short wavelength limit. In what follows, we will consider the linearized equations of motion for a plane-wave disturbances. We restrict ourselves here to 2D flows under plane-strain (4).

We will begin by writing the two-dimensional rigid-plastic equations ((7) and (8)) in non-dimensional form as follows:

$$\hat{u}_i = u_i/u_0, \quad \hat{\sigma} = \sigma/\rho g L, \quad \hat{x}_i = x_i/L, \quad \hat{t} = t\,g/u_0, \tag{9}$$

where u_0 is some characteristic velocity and L is some characteristic length-scale of the problem. The equations for the rigid-plastic flow then become

$$\frac{\partial \hat{u}_i}{\partial \hat{t}} + \mathrm{Fr}^2\hat{\boldsymbol{u}}\cdot\hat{\nabla}u_i = g_i/g - \hat{\nabla}_j\left(\hat{\sigma}\left(\delta_{ij} - kA_{ij}\right)\right), \tag{10}$$

$$\hat{\nabla}\cdot\hat{\boldsymbol{u}} = 0, \tag{11}$$

where $\mathrm{Fr} = u_0/\sqrt{gL}$ is the Froude number. For the remainder of this section we will drop the $\hat{}$ notation and assume that we are dealing with dimensionless quantities.

The linearized equations of motion for a small disturbance $(\delta\mathbf{u}, \delta\sigma)$ propagating on a smooth background solution $(\mathbf{u}, \sigma)$ to Equations (10) and (11) can be shown to be

$$\frac{\partial \delta u_i}{\partial t} + \mathrm{Fr}^2\left(\boldsymbol{u}\cdot\nabla\delta u_i + \delta\boldsymbol{u}\cdot\nabla u_i\right) = \nabla_j\left(\left(kA_{ij} - \delta_{ij}\right)\delta\sigma + \sigma\,\delta A_{ij}\right), \tag{12}$$

$$\nabla\cdot\delta\boldsymbol{u} = 0, \tag{13}$$

where

$$\delta A_{ij} = \hat{A}_{ijkl}\frac{\nabla_k \delta u_l}{\|V\|} = \left(\delta_{i(k}\delta_{l)j} - A_{ij}A_{kl}\right)\frac{\nabla_k \delta u_l}{\|V\|}.$$

We now consider plane-wave disturbances $(\delta \boldsymbol{u}, \delta p) = \exp(\lambda t + \mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x})(\boldsymbol{a}, \alpha)$ propagating with wavevector ξ. In general α and $\boldsymbol{a}$ will be complex quantities. From the linearized equations, we obtain the following relations for λ, $\boldsymbol{a}$ and α:

$$\lambda a_i = B_i \alpha + C_{ij} a_j, \tag{14}$$

$$\boldsymbol{\xi}\cdot\boldsymbol{a} = 0, \tag{15}$$

where

$$B_i = \left(k\nabla_j A_{ij} + \mathrm{i}\left(kA_{ij} - \delta_{ij}\right)\xi_j\right), \tag{16}$$

$$C_{ij} = -\mathrm{Fr}^2\left(\nabla_j u_i + \mathrm{i}(\boldsymbol{\xi}\cdot\boldsymbol{u})\delta_{ij}\right) + \xi_l\left(-\mu\hat{A}_{ijkl}\xi_k + \mathrm{i}\nabla_k\left(\mu\hat{A}_{ijkl}\right)\right). \tag{17}$$

One can solve (14) and (15) for λ for every wavevector ξ. The real part of λ determines the growth or decay of the plane-wave disturbance with wavevector ξ, and the imaginary part determines the speed of propagation of the disturbance. If the real part of λ is positive for any ξ, we refer to this mode as *linearly unstable*, as this mode will grow rapidly in time. If, for a unique background solution of the equations, there are unstable modes with growth rates $\mathfrak{Re}(\lambda) > 0$ that diverge in the short-wavelength limit ($\xi \to \infty$) then we will call these equations and the corresponding solutions *linearly ill-posed*.

Using the condition $\boldsymbol{\xi}\cdot\boldsymbol{a} = 0$ we can eliminate α from the equation for λ:

$$\lambda a_i = P_{ij} a_j, \tag{18}$$

$$P_{ij} = \left(\delta_{ik} - \frac{B_i \chi_k}{\mathbf{B}\cdot\boldsymbol{\chi}}\right) C_{kj}, \tag{19}$$

$$\alpha = -\frac{C_{ij}}{\mathbf{B}\cdot\boldsymbol{\chi}}\chi_i a_j, \tag{20}$$

where $\boldsymbol{\chi} = \boldsymbol{\xi}/|\xi|$. The eigenvalues of the matrix P (19) determine the growth and propagation of the infinitesimal plane-wave disturbance.

Since $\left(\delta_{ij} - \frac{B_i \chi_j}{\mathbf{B}\cdot\boldsymbol{\chi}}\right)B_j = 0$, it follows that P_{ij} has at least one zero eigenvalue $\lambda_1 = 0$. Since we are working in two dimensions, the remaining eigenvalue is equal to the trace of P: $\lambda_2 = \mathrm{Tr}(P)$. This eigenvalue is thus given by

$$\lambda_2 = \mathrm{Tr}(C) - \frac{C_{ij}\chi_i B_j}{\mathbf{B}\cdot\boldsymbol{\chi}}. \tag{21}$$

We now consider the short wavelength ($|\xi| \to \infty$) limit of (21). The leading-order term in ξ on the right-hand side goes as $\mathrm{O}(|\xi|^2)$ and is real with coefficient

$$-\mu\frac{A(A - k/2)}{1 - kA}, \tag{22}$$

where $A = \chi_i A_{ij}\chi_j$. This leading order term was considered by Schaeffer [5] in his analysis of (7) and (8). In two dimensions it is straight-forward to show that $|A_{ij}\chi_j|^2 = 1/2$ so that $|A|^2 = |\chi_i A_{ij}\chi_j|^2 \le 1/2$. Thus the denominator is always positive for angles of friction $\varphi < \pi/2$. However we see that the numerator is negative for $0 < A < \frac{k}{2}$. This occurs when the direction of χ lies between the direction of the velocity characteristics (these lie at $\pm\pi/4$ to the principal stress directions) and the direction of the stress characteristics (these lie at angles of $(\pm\varphi+\pi/2)/2$ to the principal stress directions) of the background solution.

We conclude that plane wave disturbances with wavevectors ξ that lie in directions between the stress and velocity characteristic directions will be unstable in the short wavelength limit as the real part of λ_2 will be positive and is $O(\xi^2)$ in this limit. Thus we conclude that the two-dimensional granular flow equations are linearly ill-posed [5].

3. Granular temperature

Our aim is to introduce microstructural terms in order to regularize the ill-posedness of the 2D IRPF equations. To do this we now introduce a granular temperature by decomposing the velocity field into a mean, slowly varying part $\mathbf{U}(\boldsymbol{x},t)$ and a rapidly fluctuating part $\boldsymbol{u}'(\boldsymbol{x},t)$:

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{U}(\boldsymbol{x},t)+\boldsymbol{u}'(\boldsymbol{x},t). \tag{23}$$

Taking the time-average of $\boldsymbol{u}(\boldsymbol{x},t)$ then yields

$$\langle\boldsymbol{u}(\boldsymbol{x},t)\rangle=\boldsymbol{U}(\boldsymbol{x},t), \tag{24}$$

so that $\langle\boldsymbol{u}'\rangle=0$. If we then let

$$T_{ij}=\langle u_i'u_j'\rangle, \tag{25}$$

then the granular temperature is defined by $T=\mathrm{Tr}(T_{ij})$. The fluctuating velocity field introduces a new scale which we write as $T_0^{1/2}$. Thus, the granular temperature scales as T_0. In what follows we will assume that $T_{ij}=\frac{1}{n}T\delta_{ij}$ where n is the number of spatial dimensions.

The incompressibility condition can now be split into a mean and fluctuating part. By 'fluctuating part' we mean the difference between the full Equation (1) and the equation obtained by averaging (1):

$$\nabla\cdot\boldsymbol{U}=0, \tag{26}$$

giving the fluctuating part as

$$\nabla\cdot\boldsymbol{u}'=0. \tag{27}$$

Now consider the fluctuating part of the momentum equation (2). This averaged momentum equation is given by

$$\rho\left(\frac{\partial U_i}{\partial t}+\boldsymbol{U}\cdot\nabla U_i\right)+\nabla_j\left(\langle\sigma_{ij}\rangle+\rho T_{ij}\right)=\rho g_i. \tag{28}$$

Subtracting this from (2) gives the so-called fluctuating part of the momentum equation:

$$\rho\left(\frac{\partial u_i'}{\partial t}+\boldsymbol{U}\cdot\nabla u_i'+u_j'\nabla_j U_i+u_j'\nabla_j u_i'\right)+\nabla_j\sigma_{ij}=\nabla_j\left(\langle\sigma_{ij}\rangle+\rho T_{ij}\right), \tag{29}$$

where we have made use of (1) and (23). Now multiplying (29) by u_k' and contracting over the indices i and k, we obtain an equation for the granular temperature:

$$\frac{\rho}{2}\left(\frac{\partial T}{\partial t}+\boldsymbol{U}\cdot\nabla T+\nabla_j\langle u_i'u_i'u_j'\rangle\right)+\rho\,T_{ij}\nabla_j U_i=-\langle u_i'\nabla_j\sigma_{ij}\rangle, \tag{30}$$

A conservation equation for the granular temperature can be developed on general grounds [12]:

$$\frac{\rho}{2}\left(\frac{\partial T}{\partial t}+\boldsymbol{U}\cdot\nabla T+U_i T_{ij}\nabla_j+\frac{1}{2}\nabla_j\langle u_i' u_i' u_j'\rangle\right)$$
$$=-\nabla_j Q_j-\langle\sigma_{ij}\rangle\nabla_i U_j-\gamma, \tag{31}$$

where $\boldsymbol{Q}$ is the energy flux vector and γ is the rate of energy dissipation per unit volume. This form of conservation equation for granular temperature can also be obtained from the kinetic theory of inelastic spheres [11]. Bocquet *et al.* [14] applied this kinetic theory to dense Couette flows, deriving the following expressions for Q_i and γ in the dense limit of the kinetic theory:

$$Q_i=-\kappa\frac{\sigma d}{T^{1/2}}\nabla_i T, \tag{32}$$

$$\gamma=\epsilon\frac{\sigma T^{1/2}}{d}, \tag{33}$$

where $\kappa>0$ and $\epsilon>0$ are dimensionless material constants, and d is a representative grain diameter. In what follows we will use the conservation equation for temperature (31) with the energy flux and dissipation terms given by (32) and (33).

Comparing Equation (30) then with (31), we arrive at the following identity:

$$\langle\sigma_{ij}V_{ij}\rangle+\epsilon\frac{\sigma T^{1/2}}{d}=\nabla_i\left(\langle\sigma_{ij}u_j'\rangle+\kappa\frac{\sigma d}{T^{1/2}}\nabla_i T\right). \tag{34}$$

It is useful now to work with non-dimensional quantities. Writing $D=d/L$, we find that

$$\langle\sigma_{ij}V_{ij}\rangle+\frac{1}{D}\left(\frac{T_0^{1/2}}{u_0}\right)\epsilon\sigma T^{1/2}=\left(\frac{T_0^{1/2}}{u_0}\right)\nabla_i\left(D\kappa\frac{\sigma}{T^{1/2}}\nabla_i T+\left(\frac{T_0^{1/2}}{u_0}\right)\langle\sigma_{ij}u_j'\rangle\right). \tag{35}$$

Now, for smooth flows we expect $D\ll 1$. In this case, since the first term on the lefthand side is O(1), we find the following relation for the temperature scale T_0 must hold:

$$T_0^{1/2}\sim D\,u_0, \tag{36}$$

and thus $T_0\ll u_0^2$.

We now consider the fluctuating part of the stress tensor. Returning to the fluctuating part of the momentum equation we can estimate the relative magnitude of the stress fluctuations, $\Delta\sim\sigma'/\rho g L$, that correspond to the velocity fluctuations $u'\sim T_0^{1/2}$. Rewriting (29) in terms of non-dimensional quantities and rearranging in terms of the stress fluctuations $\sigma_{ij}-\langle\sigma_{ij}\rangle$ we find:

$$\frac{\Delta}{D}\nabla_j\left(\sigma_{ij}-\langle\sigma_{ij}\rangle\right)=\frac{\partial u_i'}{\partial t}+\mathrm{Fr}^2\left(U_j\nabla_j u_i'+u_j'\nabla_j U_i\right)+D\mathrm{Fr}\nabla_j\left(u_j'u_i'-T_{ij}\right), \tag{37}$$

where we have made use of (36) to eliminate T_0. We will only consider slow, smooth flows here where $\mathrm{Fr}\ll 1$ and $D\ll 1$, so we see that in this case $\Delta\sim D$. Thus for slow smooth flows, we expect stress fluctuations of order $\rho g d$ to couple to the fluctuating part of the velocity.

This analysis has given us some indication of how the granular temperature and stress fluctuations may scale in smooth slow flows by appealing to the dense-limit of the kinetic theory of inelastic gases. As yet we have not specified the stress tensor nor the nature of the stress fluctuations. This is our goal in the next section.

4. Mean stress

To relate the velocity and stress fluctuations, we now assume that the flow rule (5) and constitutive law (6) hold for the full stress and velocity fields:

$$\sigma_{ij} = \sigma \left(\delta_{ij} - k A_{ij} \right). \tag{38}$$

We can decompose this stress tensor into average and fluctuating parts by expanding σ and $\mathbf{u}$ about their means, $\langle \sigma \rangle = \bar{\sigma}$ and $\langle \mathbf{u} \rangle = \mathbf{U}$ respectively. To facilitate this we introduce the following notation:

$$\begin{aligned} \bar{V}_{ij} &= V_{ij}|_{\boldsymbol{u}=\boldsymbol{U}} = \nabla_{(i} U_{j)}, \\ \bar{\sigma}_{ij} &= \bar{\sigma} \left(\delta_{ij} - k \bar{A}_{ij} \right), \\ &\ldots \text{etc.}, \end{aligned}$$

so that the bar denotes evaluation at $\boldsymbol{u} = \boldsymbol{U}$. Now, expanding σ_{ij} as indicated, where σ' is the fluctuating part of σ, and continuing to work with non-dimensional quantities, we obtain

$$\begin{aligned} \sigma_{ij} = \bar{\sigma}_{ij} &+ D \left(\sigma' \left(\delta_{ij} - k \frac{\bar{V}_{ij}}{\|\bar{V}\|} \right) - \frac{k\sigma}{\|\bar{V}\|} \hat{A}_{ijkl} \nabla_k u'_l \right) \\ &- D^2 \frac{k}{\|\bar{V}\|} \left(\hat{A}_{ijkl} \sigma' - \frac{\bar{\sigma}}{\|\bar{V}\|} \hat{C}_{ijklmn} \nabla_m u'_n \right) \nabla_k u'_l + \mathrm{O}(D^3), \end{aligned} \tag{39}$$

where

$$\hat{A}_{ijkl} = \delta_{i(k} \delta_{l)j} - \bar{A}_{ij} \bar{A}_{kl}, \tag{40}$$

and

$$\hat{C}_{ijklmn} = \frac{1}{2} \left(\hat{A}_{ijkl} \bar{A}_{mn} + \hat{A}_{klmn} \bar{A}_{ij} + \hat{A}_{mnij} \bar{A}_{kl} \right). \tag{41}$$

Note that we have assumed that we are dealing with a smooth, slow flow so that (36) holds. Consequently this expansion produces a power series in D. Further, we can now compute the mean stress

$$\langle \sigma_{ij} \rangle = \bar{\sigma}_{ij} - \frac{kD^2}{\|\bar{V}\|} \left(\hat{A}_{ijkl} \langle \sigma' \nabla_k u'_l \rangle - \frac{\bar{\sigma}}{\|\bar{V}\|} \hat{C}_{ijklmn} \langle \nabla_k u'_l \nabla_m u'_n \rangle \right), \tag{42}$$

to order D^2. Note that the $\mathrm{O}(D)$ terms have vanished as they are linear in the (zero mean) fluctuating variables. Thus the mean stress $\langle \sigma_{ij} \rangle$, which appears in the averaged momentum equation and the temperature conservation equation, receives contributions of order D^2 due to variances in velocity and stress fluctuations. Specifically these $\mathrm{O}(D^2)$ contributions to the average stress involve the variances $\langle \sigma' \nabla_k u'_l \rangle$ and $\langle \nabla_k u'_l \nabla_m u'_n \rangle$. To close this system, we wish to write these variances as functions of the mean variables $(\boldsymbol{U}, \bar{\sigma}, T)$. Here we propose the following simple relations:

$$\langle \sigma' \nabla_k u'_l \rangle = \psi \nabla_k \left(T^{1/2} \nabla_l \bar{\sigma} \right), \tag{43}$$

$$\langle \nabla_k u'_l \nabla_m u'_n \rangle = 2\phi \nabla_k \nabla_m T_{ln} = \phi \delta_{ln} \nabla_k \nabla_m T, \tag{44}$$

where ψ and ϕ are dimensionless constants. In the next section we will motivate this choice of closure relations by showing they lead to a well-posed system of equations for smooth, slow two-dimensional flows.

The average stress now takes the form:

$$\langle\sigma_{ij}\rangle = \bar{\sigma}_{ij} - D^2\frac{k}{\|\bar{V}\|}\left(\psi\hat{A}_{ijkl}\nabla_k(T^{1/2}\nabla_l\bar{\sigma}) - \frac{\phi\bar{\sigma}}{\|\bar{V}\|}\delta_{k(i}\bar{A}_{j)m}\nabla_k\nabla_m T\right). \tag{45}$$

We can now write down a set of equations for incompressible rigid-plastic flow involving only averaged quantities, correct to $\mathrm{O}(D^2)$:

$$\nabla\cdot\boldsymbol{U} = 0, \tag{46}$$

$$\begin{aligned}&\left(\frac{\partial U_i}{\partial t} + \mathrm{Fr}^2\left(\boldsymbol{U}\cdot\nabla U_i + D^2\nabla_j T_{ij}\right)\right) + \nabla_j\left(\bar{\sigma}(\delta_{ij} - k\bar{A}_{ij})\right)\\ &\quad - kD^2\nabla_j\left(\frac{\psi}{\|\bar{V}\|}\hat{A}_{ijkl}\nabla_k(T^{1/2}\nabla_l\bar{\sigma}) - \frac{\phi\bar{\sigma}}{\|\bar{V}\|^2}\delta_{k(i}\bar{A}_{j)m}\nabla_k\nabla_m T\right) = g_i/g,\end{aligned} \tag{47}$$

$$\begin{aligned}&\frac{D^2}{2}\left(\frac{\partial T}{\partial t} + \mathrm{Fr}^2\left(U_j\nabla_j T + T_{ij}\nabla_j U_i\right)\right)\\ &\quad = D^2\left(\kappa\nabla_j\left(\frac{\bar{\sigma}}{T^{1/2}}\nabla_j T\right) - \frac{k\phi\bar{\sigma}}{\|\bar{V}\|}\nabla^2 T\right) + \bar{\sigma}\left(k\|\bar{V}\| - \epsilon T^{1/2}\right).\end{aligned} \tag{48}$$

These constitute a closed set of equations for the averaged variables $\boldsymbol{U}$, $\bar{\sigma}$ and T.

5. Stability

In this section we examine the stability of slow, smooth solutions (*i.e.,* $\mathrm{Fr} \ll 1$, $D \ll 1$) to Equations (48–50). Again we consider plane-wave disturbances $(\delta\boldsymbol{u}, \delta p, \delta T) = \exp(\lambda t + \mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x})$ $(\boldsymbol{a}, \alpha, \tau)$ propagating with wavevector $\boldsymbol{\xi}$. From the linearized equations, we obtain the following equations for $\boldsymbol{a}$, α and τ:

$$\begin{pmatrix}\xi^2\mu Q_{ij} + \lambda\delta_{ij} & \mathrm{i}\xi^3 D^2 R r_i & -\mathrm{i}\xi^3 D^2 S s_i\\ \chi_j & 0 & 0\\ -\mathrm{i}\xi k\sigma\chi_i A_{ij} & -\mathrm{i}\xi D^2 G & D^2(\xi^2 H + \lambda)\end{pmatrix}\begin{pmatrix}a_j\\ \alpha\\ \tau\end{pmatrix} = 0, \tag{49}$$

where

$$\begin{aligned}&Q_{ij} = \hat{A}_{ijkl}\chi_k\chi_l + \mathrm{O}(1/\xi), \quad R = \frac{k\psi}{\|V\|}, \quad S = \mu\phi,\\ &\quad r_i = \hat{A}_{ijkl}\chi_j\chi_k\chi_l + \mathrm{O}(1/\xi), \quad s_i = \delta_{k(i}A_{j)m}\chi_j\chi_k\chi_m + \mathrm{O}(1/\xi),\\ &\quad G = \frac{\boldsymbol{\chi}\cdot\nabla T}{T^{1/2}} + \mathrm{O}(1/\xi), \quad H = \frac{\kappa\bar{\sigma}}{T^{1/2}} - \frac{k\phi}{\|\bar{V}\|} + \mathrm{O}(1/\xi).\end{aligned} \tag{50}$$

It is straightforward to eliminate α and τ to obtain the following eigenvalue problem:

$$\left(W\lambda\delta_{ik} + P_{ij}Z_{jk}\right)a_k = 0, \tag{51}$$

where

$$W = D^2(R(\lambda + \chi^2 H)r_i - \mathrm{i}\xi G S s_i)\chi_i, \tag{52}$$

$$P_{ij} = \delta_{ij} - \frac{r_i\chi_j}{(\boldsymbol{r}\cdot\boldsymbol{\chi})}, \tag{53}$$

$$Z_{jk} = \xi^2\mu W Q_{jk} - \mathrm{i}\xi^3 D^2 S s_j\chi_l(\mu G Q_{lk} - \mathrm{i}\xi k\sigma R\boldsymbol{r}\cdot\boldsymbol{\chi}A_{lk}). \tag{54}$$

Again one can show that the matrix $P_{ij}Z_{jk}$ has at least one zero eigenvalue (since $P_{ij}r_j = 0$). Thus, in two dimensions the remaining eigenvalue is equal to the trace of $P_{ij}Z_{jk}$. This identity results in a quadratic equation for this eigenvalue λ_2, giving two solutions to leading order in ξ (provided $\psi \neq 0$):

$$\lambda_2 = \frac{\xi^2\bar{\sigma}}{2}\left(-\Omega_1 \pm \sqrt{\Omega_1^2 - 4\Omega_2} + \mathrm{O}\left(\frac{1}{\xi}\right)\right), \tag{55}$$

where

$$\Omega_1 = \frac{\kappa}{T^{1/2}} + \frac{k}{\|\bar{V}\|}\left(\frac{A^2}{1-A^2} - \phi\right), \tag{56}$$

$$\Omega_2 = \frac{1}{2(1-A^2)\|\bar{V}\|}\left(\frac{\kappa}{T^{1/2}} - \frac{k\phi}{2\|\bar{V}\|}(1 + A^2 - 2A^4)\right). \tag{57}$$

If we choose $\phi < 0$, then we see that both Ω_1 and Ω_2 are positive for $k, \kappa > 0$ (recall that $|A|^2 \leq 1/2$). There are then two cases: if $\Omega_1^2 - 4\Omega_2 > 0$, then the $\mathrm{O}(\xi^2)$ contribution to λ_2 is real and since $\Omega_2 > 0$, both of the values for λ_2 are negative. However, if $\Omega_1^2 - 4\Omega_2 \leq 0$, then the real part of the $\mathrm{O}(\xi^2)$ contribution to λ_2 is $-\Omega_1\xi^2/2 < 0$. We conclude that the real part of the leading order contribution in ξ to λ_2 is negative for $\phi < 0$. Thus, smooth, slow flows governed by Equations (48)–(50) are well-posed provided we choose $\phi < 0$ and $\psi \neq 0$.

6. Discussion

Equations (48)–(50) give a set of equations for the average variables $\mathbf{U}$, $\bar{\sigma}$ and T. In the limit where $d \to 0$, these equations reduce to the equations for incompressible rigid-plastic flow plus an algebraic relation linking granular temperature and the deformation rate $\|\bar{V}\|$. We have shown that the choice of closure relations (45) and (46) lead to a well-posed set of equations for smooth slow flows under the conditions that $\phi < 0$ and $\psi \neq 0$. Physically, this first requirement corresponds to the condition that the $\nabla^2 T$ term in Equation (50) be dissipative. Thus the effect of this term in the temperature conservation equation (50) is to conduct granular temperature from hot regions to cool regions. In addition, the coefficient of the first closure relation (45) must be non-zero to preserve the coupling between high-order derivatives of the pressure and temperature in the momentum equation and guarantee well-posedness. This term in the momentum equation conserves temperature intrinsically and so does not appear in the temperature conservation equation.

Although we have not provided a clear physical motivation of the closure relations, they are dimensionally and tensorially consistent. A detailed theory of velocity and stress fluctuations could be developed to extend or replace equations (45) and (46). Another approach might be to expand the variances in all dimensionally and tensorially consistent terms involving T and $\bar{\sigma}$ and their derivatives. In this case, we suggest that demanding that the resulting equations be well-posed would be a useful discrimant to apply to such closure relations. We have worked in two-dimensional plane-strain conditions here for simplicity. A full theory would need to address these issues in three dimensions for compressible granular flows.

We have also not examined the question of whether solutions to (48–50) reduce to solutions of the IRPF in this limit. Note that the equations (48–50) involve T and derivatives of T, which require new boundary conditions over and above those required by the IRPF equations. Boundary conditions that are placed on the granular temperature will effectively place new conditions on the velocity derivatives through Equation (50) in the limit $d \to 0$. The nature of these boundary conditions on the temperature will determine whether solutions to (48)–(50) reduce to solutions of the IRPF as $d \to 0$. For instance, one could demand that any boundary conditions on T preserve solutions of the IRPF equations in this limit. Whether this is physically reasonable or not, requires a more detailed examination of granular temperature than we have undertaken here.

An important assumption in deriving the expression for the average stress (41), was that the full stress tensor obeyed the constitutive laws (5) and (6). This allowed us to relate velocity fluctuations to the stress fluctuations. Savage [12] relates velocity and stress fluctuations using a constitutive law developed directly from the kinetic theory of inelastic gases. Savage also suggests a relationship between the temperature scale and the particle size that is very similar to (36). Indeed, our analysis in Section 3 can be directly applied to the equations of Savage.

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